PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2890. Proposed by B. F. FINKEL, Drury College.

Having given a triangle whose base is $2c$ and (*a*) the sum of whose other two sides is $2a$, (*b*) the difference of whose other two sides is $2a$, determine the envelope of the perpendicular bisectors of the variable sides.

2891. Proposed by D. F. BARROW, Philomath, Ga.

Let A', A'', A''', and P denote, respectively, the vertices of a triangle and any point in its plane; and let P', P'', P''', denote the feet of the perpendiculars from P upon the sides opposite A', A'', A'''. Now suppose each of the lines PP', PP'', PP''' to revolve about P through an angle α; and let P_α', P_α'', P_α''' denote the intersections of this new triad of lines with the corresponding sides of the triangle. As α varies, find the envelope of the variable circle through P_α', P_α'', P_α'''

2892. Proposed by R. T. McGREGOR, Bangor, Calif.

Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

2893. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two given skew lines in a variable plane turning about a fixed axis, not coplanar with either of the given lines.

2894. Proposed by PHILIP FRANKLIN AND E. L. POST, Princeton University.

Given the following set of assumptions concerning a set S and certain undefined sub-classes of S, called m-classes:

I. If A and B are distinct elements of S, there is at least one m-class containing both A and B.

II. If A and B are distinct elements of S, there is not more than one m-class containing both A and B.

Def. Two m-classes with no elements in common are called *conjugates.*

III. For every m-class there is at least one *conjugate* m-class.

IV. For every m-class there is not more than one conjugate m-class.

V. There exists at least one m-class.

VI. Every m-class contains at least one element of S.

VII. Every m-class contains not more than a finite number of elements.

Develop some of the propositions of the "mathematical science" (cf. Veblen and Young, *Projective Geometry,* Vol. I, pp. 1 f.) based on them and in particular develop a sufficient number of theorems to prove that the set of assumptions is categorical and give a concrete representation of the set S which satisfies them. Also prove that the assumptions are independent.

2895. Proposed by R. M. MATHEWS, Wesleyan University.

To construct an equilateral triangle with its vertices lying on: (*a*) any three coplanar lines; (*b*) three parallels in space; and (*c*) any three lines in space.

PROBLEMS—NOTES

10. A Curve of Pursuit. The extended discussion of a curve of pursuit in a recent issue of this MONTHLY (*1921,* 54–61, 91–97) suggests this note. In *Nouvelle Correspondance Mathêmatique,* volume 3, 1877, E. Lucas proposed the following problem in May (pages 175–176): "Three dogs are placed at the vertices of an equilateral triangle; they run one after the other. What is the curve described

by each of them?" In the issue for August, 1877, H. Brocard gave (page 280) the following result: Supposing the dogs start at the same time and with the same velocity, the curve of pursuit of each of the dogs is a logarithmic spiral, having for pole the center of the triangle, and tangent at a vertex of the triangle to one of its sides. ARC.

11. In *Revista Matemática Hispano-Americana*, September, 1920, the following problem is solved (pages 228–229): "Construct a square knowing the points of intersection of its sides with a line of its plane;" three solutions are found. The more general problem: To describe a square circumscribing a given quadrilateral, has been discussed many times since Diesterweg's solution in 1828.[1] There are six solutions, as Lehmus remarked in 1847,[2] and these are illustrated by a figure in I. Ghersi, *Matematica dilettevole e curiosa*, Milano, 1913, p. 587. The problem was also discussed by T. Clausen.[3] If the diagonals of the quadrilateral are equal and orthogonal there is an infinite number of solutions; this result is a particular case of a theorem given by J. Murènt in *Nouvelles Annales de Mathématiques*, 1855, p. 365: The necessary and sufficient condition that it is possible to circumscribe to a given quadrilateral an infinite number of rectangles, similar to a given rectangle, is that the diagonals of the quadrilateral shall be at right angles to one another and proportional to the sides of the given rectangle.[4] Hence, when diagonals so related are unequal, there must be a finite number of solutions of the square problem.

But Diesterweg's problem is only a particular case of a problem discussed by Lamé in his *Examen des différentes Méthodes employées pour résoudre les Problèmes de Géométrie*, Paris, 1818, pp. 16–17: About a given quadrilateral describe another similar to a third quadrilateral. He points out that there are, in general, eight solutions, closely allied to those of the inverse problem: To inscribe in a given quadrilateral another similar to a third quadrilateral. Numerous discussions of these problems are to be found in periodicals and books. The construction of a square inscribed in a quadrilatered was discussed analytically by Carnot[5] in 1803, and he stated that there were three solutions in general. T. Clausen in 1864 showed[6] that the problem had, in general, six solutions, and he commented on dual relations connecting it with the problem which we have traced to Diesterweg. ARC.

12. Milner's Lamp. In *The Journal of the Indian Mathematical Society*, June, 1920, the following problem is proposed for solution, on page 119, by A. Narasinga Rao: "Determine generally the form of a vessel whose contents are

[1] W. A. Diesterweg, *Geometrische Aufgaben nach der Methode der Griechen*. Andere Sammlung, Elberfeld, 1828, pp. 172–173.

[2] *Journal für die reine und augewandte Mathematik*, vol. 35, p. 281.

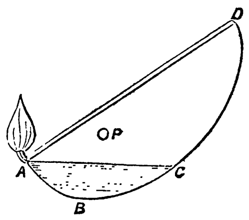
[3] *Archiv der Mathematik und Physik*, vol. 15, 1850, pp. 238–239.

[4] C. M. Herbert seemed to regard the special case of this theorem for the square as new, in his articles in *Annals of Mathematics*, second series, vol. 16, 1914, pp. 42 and 67.

[5] Carnot, *Géométrie de Position*, pp. 374–377.

[6] *Bulletin de l'acad. imp. d. sc. de St. Pétersbourg*, vol. 7, 1864, cols. 177–181.

just spilling over in the position of equilibrium, whatever the amount of liquid it contains (1) when it rests on a horizontal plane, (2) when it is suspended about a horizontal axis." This reminds one of problem number 2353 proposed by De Morgan in *The Educational Times* about fifty-five years ago: "The late Dr. Milner, President of Queen's College, Cambridge, constructed a lamp, which General Perronet Thompson remembers to have seen. It is a thin cylindrical bowl, revolving about an axis at P, and the curve $ABCD$ is such that, whatever quantity of oil ABC may be in the bowl, the position of equilibrium is such that the oil just wets the wick at A. What is the curve $ABCD$?" (Cf. *Mathematical Questions with their Solutions from the Educational Times*, volume 7, 1867, p. xvi. A few years later De Morgan referred to the problem in his *A Budget of Paradoxes*, London, 1872, p. 149; second edition by D. E. Smith, 1915, vol. 1, p. 252.) A solution by D. Biddle was published in *Mathematical Questions* . . ., volume 49, 1888, pp. 54–55. He found that a very near approach to the curve required was $r = \cos^{1/2}\theta$. This is one of a family of curves $r = a\cos^n m\theta$, arising in applications of descriptive geometry (cf. Gabriel Marie, *Exercices de Géométrie Descriptive,* 4e éd. Tours, 1909, pp. 835–842; indeed the special case $r=a\cos^{1/2}\theta$ is discussed on page 841).



The problem of the curve for Milner's lamp was considered by Tait, who refers to its formulation in De Morgan's *Budget,* in a paper read before the Edinburgh Mathematical Society in 1887.[1] He quoted De Morgan's statement that the lamp was "a hollow-semi-cylinder, but not with a circular curve," and arrived at a "direct contradiction" of this statement. As a question in connection with the differential equation caused trouble he applied to Cayley who in reply showed,[2] that starting with Tait's differential equation, the solution found was correct. Without any reference to Tait, Biddle discussed the circular form and the consequent lack of "bias to cause rotation." ARC.

PROBLEMS—SOLUTIONS

2799 [1920, 31]. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

A newspaper recently gave this problem: Cut a regular six-pointed star into the fewest number of pieces which will fit together and make a square. The newspaper gave a solution in seven pieces. First cut off two opposite points of the star. Divide each into two parts, and fit to the remaining portion of the star so as to make a rectangle. Find the mean proportional between the length and breadth of this rectangle (construction not shown); this is the side of the required square. Using this dimension on the two long sides of the rectangle, divide the latter into three pieces, which make the square. Total seven pieces.

How may the square be formed with not more than five pieces?

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[1] P. G. Tait, "Note on Milner's lamp," *Proceedings of the Edinburgh Mathematical Society,* vol. 5, 1887, pp. 97–98; *Scientific Papers by Peter Guthrie Tait,* vol. 1, 1900, pp. 215–216.

[2] A. Cayley, "On a differential equation and the construction of Milner's lamp," *Proceedings of the Edinburgh Mathematical Society,* vol. 5, 1887, pp. 99–101; *Collected Mathematical Papers of Arthur Cayley,* vol. 13, 1897, pp. 3–5.